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A Numerical Method for Solving the Dirichlet Problem for the Wave Equation

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Abstract—In this paper we present a numerical method for solving the Dirichlet problem for a two-dimensional wave equation. We analyze the ill-posedness of the problem and construct a regularization algorithm. Using the Fourier series expansion with respect to one variable, we reduce the problem to a sequence of Dirichlet problems for one-dimensional wave equations. The first stage of regularization consists in selecting a finite number of problems from this sequence. Each of the selected Dirichlet problems is formulated as an inverse problem $Aq = f$ with respect to a direct (well-posed) problem. We derive formulas for singular values of the operator A in the case of constant coefficients and analyze their behavior to judge the degree of ill-posedness of the corresponding problem. The problem $Aq = f$ on a uniform grid is reduced to a system of linear algebraic equations $A_{ll}q = F$. Using the singular value decomposition, we find singular values of the matrix A_{ll} and develop a numerical algorithm for constructing the r-solution of the original problem. This algorithm was tested on a discrete problem with relatively small number of grid nodes. To improve the calculated r-solution, we applied optimization but observed no noticeable changes. The results of computational experiments are illustrated.

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INTRODUCTION

The first results concerning the boundary value problems for hyperbolic equations with data on the whole boundary were obtained by J. Hadamard [1], A. Huber [2], D. Mangeron [3].

D. G. Bourgin and R. Daffin [4, 5] studied the Dirichlet problem in the rectangle

$$R := \{0 \le t \le T, 0 \le x \le S\}$$

for the damped wave equation $\left(D_t^2 - D_x^2 - k^2\right)u = 0$. From here on $D_t u = \dfrac{\partial u}{\partial t}$, $D_x u = \dfrac{\partial u}{\partial x}$, $D_t^2 u = \dfrac{\partial^2 u}{\partial t^2}$, etc. If T/S is irrational then the problem was shown to admit at most one solution in the class of continuously differentiable functions that have the Lebesgue integrable second derivatives in R. The authors also proved the solvability of the Dirichlet problem under certain restrictions on the parameters T, S, and k.

S. G. Ovsepyan [6] considered the problem

$$(1 + \lambda)D_x^2 u - (1 - \lambda)D_y^2 u = 0, \qquad u|_\Gamma = \sigma(s), \tag{1}$$

in a bounded multiply connected domain D with boundary Γ (here λ is a real parameter, $|\lambda| < 1$). It was demonstrated that, under some conditions on the boundary Γ, the solution to problem (1) is unique in

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the first Baire class. The solution of problem (1) was also shown as unstable with respect to a change of domain not only for simply connected domains, but also for any bounded multiply connected domain D with piecewise smooth boundary Γ.

Yu. M. Berezanskii [7] considered the Dirichlet problem for the equation

$$\big(D_t^2 - D_x^2\big)u = 0, \tag{2}$$

and suggested seeking its solutions not in the class of smooth functions but in L_2. This suggestion made it possible to construct the domains in which the generalized solution of the Dirichlet problem is stable under small perturbations of boundary. Some examples of such domains were described in [7].

The studies by V. M. Borok [8–11] concern the boundary value problems in the layer $\{(t,x) : 0 \le t \le T,\ x \in \mathbb{R}^n\}$ for evolution equations and systems with constant coefficients. Classes of uniqueness and stable solvability of the problems are specified in the papers. In particular, in [8], it is shown that equation (2) with the boundary conditions $u(x,0) = u(x,T) = 0$, $-\infty < x < \infty$, has nontrivial solutions in the class of bounded functions.

The necessity of considering the boundary value problems for hyperbolic equations has also emerged from the theory of time-dependent problems for linear systems of differential equations unsolvable with respect to the higher time derivatives. This theory was initiated by S. L. Sobolev [12, 13] (see also papers by R. A. Aleksandryan [14] and R. Denchev [15]).

The results by T. I. Zelenyak and M. V. Fokin [16–18] concerning the solvability and spectral properties of the Dirichlet problem for the wave equation were stimulated by studying the asymptotic behavior of solutions to the problem

$$\frac{d^2u}{dt^2} + Au = 0, \qquad u(0) = u_0, \qquad u'(0) = u_1,$$

where A is a positive self-adjoint bounded operator in the Hilbert space $W_2^1(D)$.

S. A. Aldashev [19] proved that the Dirichlet problem for the multidimensional wave equation is uniquely solvable in cylinder.

A fairly complete list of references can be found in the monographs by B. I. Ptashnik [20] and V. P. Burskii [21].

In Section 1 of this article, we formulate direct and inverse problems for a wave equation. In Section 2, we state the conditions under which the Dirichlet problem for the wave equation is well-posed, and calculate singular values of the operator of the inverse problem with constant coefficients. In the next section, we discretize the inverse problem with variable coefficients, reduce it to a system of linear algebraic equations (SLAE), and calculate singular values of this system. Section 4 presents the results of numerical calculations.

1. STATEMENT OF THE PROBLEM

Consider the problem of evaluating the water-surface fluctuations that are caused by a sudden displacement of the seafloor occurred at time $t = 0$ and described by the finite function $f^{(1)}(x,y) = u(x,y,0)$. We assume that the shape of the water surface is fixed at time $t = T$ and has the form $f^{(2)}(x,y) = u(x,y,T)$. We suppose also that by time T the wave has not yet reached the shore; consequently, along the edge of the sea we can put the homogeneous boundary conditions. Using shallow water approximation [23], denote $c(x,y) = \sqrt{gH(x,y)}$, where $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ is acceleration of gravity, $H(x,y) > 0$ is a function describing the bottom topography (bathymetry). Thus, we come to the following Dirichlet problem for the wave equation:

$$\begin{gathered} D_t^2 u = g[D_x(H(x,y)D_x u) + D_y(H(x,y)D_y u)], \qquad (x,y) \in \Omega, \qquad t \in (0,T); \\ u|_{t=0} = f^{(1)}(x,y), \qquad u\,|_{t=T} = f^{(2)}(x,y), \qquad (x,y) \in \Omega; \qquad u|_{\partial\Omega} = 0, \qquad t \in (0,T), \end{gathered} \tag{3}$$

where $\Omega = (0,L) \times (0,L)$.

To provide the homogeneity of conditions along $\partial\Omega$, we suppose that the support of the function $f^{(1)}(x,y)$ is sufficiently small:

$$\sup(f^{(1)}) \in \Omega(a) = (L/2 - a, L/2 + a) \times (L/2 - a, L/2 + a), \qquad a \in (0, L/2),$$

and so is the parameter $T \in (0, T_{\max})$, where $T_{\max} = (L/2 - a)/\|c\|_{C(0,L)}$.

Problem (3) is ill-posed (its instability is shown in [22]). We reformulate this problem as inverse to a direct (well-posed) problem. Indeed, consider the following initial boundary value problem for the wave equation:

$$\begin{gathered} D_t^2 u = g\big[D_x(H(x,y)D_x u) + D_y(H(x,y)D_y u)\big], \qquad (x,y) \in \Omega, \quad t \in (0,T); \\ u|_{t=0} = f^{(1)}(x,y), \qquad D_t u\,|_{t=0} = q(x,y), \qquad (x,y) \in \Omega; \qquad u|_{\partial\Omega} = 0, \qquad t \in (0,T). \end{gathered} \tag{4}$$

In the direct problem (4), it is required to find the function $u(x,y,t)$ from the given $f^{(1)}(x,y)$, $q(x,y)$, and $H(x,y)$ (a method of solving this problem was proposed in [24]).

Assume $q(x,y)$ unknown. Let an additional information be gained about the solution of (4):

$$u(x,y,T) = f^{(2)}(x,y). \tag{5}$$

Then the inverse problem is formulated as follows:
Given $f^{(1)}(x,y)$, $f^{(2)}(x,y)$, and $H(x,y)$, determine $q(x,y)$ using (4) and (5).

2. STUDYING THE PROBLEM IN THE CASE OF CONSTANT COEFFICIENTS

2.1. The Uniqueness Theorem

Consider the inverse problem (4), (5) in the case when $H(x,y) = H$ and suppose that $c = \sqrt{gH} = 1$. Extend all functions involved in (4), (5) as odd functions in the variable y over the interval $(-L, L)$. Expanding them in the Fourier series like

$$u(x,y,t) = \sum_{k\in\mathbb{N}} u_k(x,t)\sin(\pi k y/L),$$

we arrive at the following sequence of inverse problems for one-dimensional wave equations:

$$D_t^2 u_k = \big(D_x^2 - k^2(\pi/L)^2\big)u_k, \qquad x \in (0,L), \qquad t \in (0,T), \tag{6}$$

$$u_k(x,0) = f_k^{(1)}(x), \qquad x \in (0,L), \tag{7}$$

$$D_t u_k(x,0) = q_k(x), \qquad x \in (0,L), \tag{8}$$

$$u_k(0,t) = u_k(L,t) = 0, \qquad t \in (0,T), \tag{9}$$

$$u_k(x,T) = f_k^{(2)}(x), \qquad x \in (0,L). \tag{10}$$

Here $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Similarly, we extend $u_k(x,t)$, $f_k^{(1)}(x)$, $f_k^{(2)}(x)$, and $q_k(x)$ into odd functions in x over the interval $(-L, L)$, then expand them in Fourier series

$$u_k(x,t) = \sum_{n\in\mathbb{N}} u_{k,n}(t)\sin(n\pi x/L),$$

and so on. In result, we obtain the inverse problems

$$u''_{k,n} + (\pi/L)^2(n^2 + k^2)u_{k,n} = 0, \qquad u_{k,n}(0) = f_{k,n}^{(1)}, \qquad u'_{k,n}(0) = q_{k,n}, \tag{11}$$

$$u_{k,n}(T) = f_{k,n}^{(2)}. \tag{12}$$

The solution of the direct problem (11) has the form

$$u_{k,n}(t) = f_{k,n}^{(1)}\cos p_{k,n}t + \frac{q_{k,n}}{p_{k,n}}\sin p_{k,n}t, \qquad p_{k,n} = \frac{\pi}{L}\sqrt{k^2 + n^2}. \tag{13}$$

Substituting $t = T$ in (13), we obtain

$$u_{k,n}(T) = f_{k,n}^{(2)} = f_{k,n}^{(1)}\cos p_{k,n}T + \frac{q_{k,n}}{p_{k,n}}\sin p_{k,n}T. \tag{14}$$

Theorem (uniqueness of the solution to the inverse problem). *Assume that, for all $k, n \in \mathbb{N}$ and $m \in \mathbb{Z}$, the parameter $T \in (0, T_{\max})$ meets the condition $T \neq \pi m/p_{k,n}$ (for example, $T = r_1/r_2$ is a rational number from the interval $(0, T_{\max})$). Then if the inverse problem (6)–(10) has a solution in $C^1[0, L]$ then the solution is unique and its Fourier coefficients are expressed by the formula*

$$q_{k,n} = \frac{f_{k,n}^{(2)} - f_{k,n}^{(1)} \cos p_{k,n} T}{\sin p_{k,n} T} \, p_{k,n}.$$

2.2. Singular Values of the Operator of the Inverse Problem

Reformulate the inverse problem (6)–(10) in the operator form $A(k)q_k = f_k^{(2)}$, where

$$A(k) : L_2(0, L) \to L_2(0, L).$$

Define $A(k)$ as follows: Take an arbitrary $q_k \in L_2(0, L)$. Substitute $q_k(x)$ into (8) and consider the direct problem (6)–(9). As known [25], if $f_k^{(1)} \in \overset{\circ}{\rightarrow} H^1(0, L)$ and $q_k \in L_2(0, L)$ then the problem (6)–(9) admits a unique solution $u_k \in H^1((0, L) \times (0, T))$. By the trace theorem [24], there exists $u_k(x, T) \in L_2(0, L)$. Put $A(k)q_k := u_k(x, T)$. Note that the operator $A(k) : L_2(0, L) \to L_2(0, L)$ so-constructed is well defined.

Theorem 2. *The singular values of $A(k)$ have the form* $\sigma_n(A(k)) = |\sin p_{k,n} T|/p_{k,n}$, $n \in \mathbb{N}$.

Proof. Taking into account (14), we can write

$$(A(k)q_k)(x) = \sum_{n \in \mathbb{N}} \frac{\sin p_{k,n} T}{p_{k,n}} \, q_{k,n} \sin \frac{\pi n x}{L}. \tag{15}$$

It is known that $\sigma_n^2(A(k)) = \lambda_n(A^*(k)A(k))$, where $\lambda_n(A^*(k)A(k))$ are the eigenvalues of the operator $A^*(k)A(k)$. Let us determine the adjoint operator $A^*(k) : L_2(0, L) \to L_2(0, L)$. By definition,

$$\langle A(k)q_k, \psi \rangle_{L_2(0,L)} = \langle q_k, A^*(k)\psi \rangle_{L_2(0,L)}, \qquad \psi(x) \in L_2(0, L). \tag{16}$$

Equality (16) can be rewritten as

$$\int_0^L (A(k)q_k)(x)\psi(x)\,dx = \int_0^L q_k(x)(A^*(k)\psi)(x)\,dx. \tag{17}$$

Expand the odd extension of ψ into the Fourier series on $(-L, L)$:

$$\psi(x) = \sum_{n \in \mathbb{N}} \psi_n \sin(\pi n x/L). \tag{18}$$

Rearrange (17) in view of (15) and (18):

$$\int_0^L \left(\sum_{n \in \mathbb{N}} \frac{\sin p_{k,n} T}{p_{k,n}} \, q_{k,n} \sin \frac{\pi n x}{L} \right) \left(\sum_{m \in \mathbb{N}} \psi_m \sin \frac{\pi m x}{L} \right) dx = \sum_{n \in \mathbb{N}} \frac{\sin p_{k,n} T}{p_{k,n}} \, q_{k,n} \psi_n$$
$$\times \int_0^L \sin^2 \frac{\pi n x}{L} dx = \int_0^L \left(\sum_{m \in \mathbb{N}} q_{k,m} \sin \frac{\pi m x}{L} \right) \left(\sum_{n \in \mathbb{N}} \frac{\sin p_{k,n} T}{p_{k,n}} \, \psi_n \sin \frac{\pi n x}{L} \right) dx.$$

Thus, we obtain

$$(A^*(k)\psi)(x) = \sum_{n \in \mathbb{N}} \frac{\sin p_{k,n} T}{p_{k,n}} \, \psi_n \sin \frac{\pi n x}{L}. \tag{19}$$

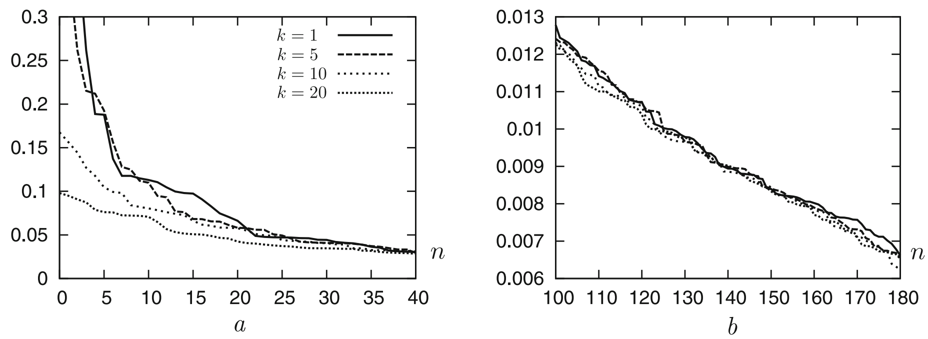


Fig. 1. Singular values $\sigma_n(A(k))$ for $k = 1, 5, 10, 20$, $T = 3$, and $c = 1$: (a) $n = 0, \dots, 40$; (b) $n = 100, \dots, 180$

Note that from (15) and (19) it follows that $A(k)$ is self-adjoint.

Using (15) and (19), we obtain the Fourier series expansion of $A^*(k)A(k)$:

$$A^*(k)(A(k)q_k)(x) = \sum_{n\in\mathbb{N}} \frac{\sin p_{k,n}T}{p_{k,n}}(A(k)q_k)_n \sin\frac{\pi n x}{L} = \sum_{n\in\mathbb{N}} \left(\frac{\sin p_{k,n}T}{p_{k,n}}\right)^2 q_{k,n} \sin\frac{\pi n x}{L}.$$

Therefore, the eigenvalues of $A^*(k)A(k)$ have the form

$$\lambda_n(A^*(k)A(k)) = \left(\frac{\sin p_{k,n}T}{p_{k,n}}\right)^2.$$

Then the singular values of $A(k)$ can be written as $\sigma_n(A(k)) = |\sin p_{k,n}T|/p_{k,n}$.

The proof of Theorem 2 is complete. □

As seen in Fig. 1, the singular values of $A(k)$ decrease with the growth of n.

3. DISCRETIZATION OF THE INVERSE PROBLEM IN THE CASE OF $c = c(x)$

In numerical calculations we consider the one-dimensional inverse problems

$$\begin{gathered} D_t^2 u_k = c^2(x)\big(D_x^2 - k^2(\pi/L)^2\big)u_k, \qquad x \in (0, L), \qquad t \in (0, T), \\ u_k(x, 0) = f_k^{(1)}(x), \qquad D_t u_k(x, 0) = q_k(x), \qquad x \in (0, L), \\ u_k(0, t) = u_k(L, t) = 0, \qquad t \in (0, T), \end{gathered} \tag{20}$$

$$u_k(x, T) = f_k^{(2)}(x), \tag{21}$$

which are obtained using the Fourier series expansions of the functions involved in the following inverse problem:

$$\begin{gathered} D_t^2 u = c^2(x)\big(D_x^2 + D_y^2\big)u, \qquad (x, y) \in \Omega, \qquad t \in (0, T), \\ u|_{t=0} = f^{(1)}(x, y), \qquad D_t u|_{t=0} = q(x, y), \qquad (x, y) \in \Omega, \qquad u|_{\partial\Omega} = 0, \qquad t \in (0, T); \end{gathered} \tag{22}$$

$$u|_{t=T} = f^{(2)}(x, y). \tag{23}$$

Note that the inverse problem (4), (5) can be investigated similarly, but it requires much more cumbersome calculations.

Let N_x be the number of nodes of a uniform grid with respect to the variable x on the interval $(0, L)$. The odd number N_t of nodes of a uniform grid with respect to the variable t is chosen so that $N_t \geq N_x$. The step in the space variable x is equal to $h_x = L/N_x$ and the time-step is $h_t = T/N_t$. Denote $r_i = c_i(h_t/h_x)$ and $a_{k_i} = (\pi/L)^2(c_i k h_t)^2/2$ for $i = 0, 1, \dots, N_x$.

Using an explicit difference scheme of the second order of accuracy, we obtain the discrete problem

$$
\begin{gathered}
u_{k_i}^{j+1} = r_i^2\big(u_{k_{i+1}}^j - 2u_{k_i}^j + u_{k_{i-1}}^j\big) - a_{k_i}\big(u_{k_{i+1}}^j + u_{k_{i-1}}^j\big) + 2u_{k_i}^j - u_{k_i}^{j-1},\\
u_{k_i}^0 = f_{k_i}^{(1)}, \qquad u_{k_i}^1 = \gamma\big(f_{k_i}^{(1)}\big) + h_t q_{k_i}, \qquad i = 0, 1, \dots, N_x,\\
u_{k_0}^j = u_{k_{N_x}}^j = 0, \qquad j = 0, 1, \dots, N_t;
\end{gathered} \tag{24}
$$

$$
u_{k_i}^{N_t} = f_{k_i}^{(2)}. \tag{25}
$$

Here $\gamma(f_{k_i}^{(1)}) = f_{k_i}^{(1)} + r_i^2\big(f_{k_{i+1}}^{(1)} - 2f_{k_i}^{(1)} + f_{k_{i-1}}^{(1)}\big)/2 - a_{k_i}\big(f_{k_{i+1}}^{(1)} + f_{k_{i-1}}^{(1)}\big)/2$.

The purpose of the next subsection is to present the inverse problem (24), (25) in the matrix form $A_{ll}(k)q_k = F_k$, where F_k is a data vector of the problem and $l = N_x + 1$.

3.1. Reducing the Inverse Problem to a System of Linear Algebraic Equations

Let $v_1 = \big(u_{k_0}^2, u_{k_1}^2, \dots, u_{k_{N_x}}^2\big)^\top$, and $v_2 = \big(u_{k_0}^3, u_{k_1}^3, \dots, u_{k_{N_x}}^3\big)^\top$.

Set $U^0 = \big(u_{k_0}^0, u_{k_1}^0, \dots, u_{k_{N_x}}^0, u_{k_0}^1, u_{k_1}^1, \dots, u_{k_{N_x}}^1\big)^\top$ and $U^1 = (v_1, v_2)^T$, $Q = (q_{k_0}, q_{k_1}, \dots, q_{k_{N_x}})^\top$. The vectors U^0 and U^1 have the same dimension equal to $2N_x + 2$. Put $b_{k_i} = r_i^2 - a_{k_i}$ for $i = 0, 1, \dots, N_x$, and note that the vectors v_1 and v_2 can be written as $v_1 = B_1 U^0$ and $v_2 = B_2 v_1 + B_3 U^0$, where

$$
B_1 = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0\\
0 & -1 & \dots & 0 & 0 & b_{k_1} & 2-2r_1^2 & b_{k_1} & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & \dots & -1 & 0 & 0 & \dots & b_{k_{N_x-1}} & 2-2r_{k_{N_x-1}}^2 & b_{k_{N_x-1}}\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix},
$$

$$
B_2 = \begin{pmatrix}
0 & 0 & 0 & \dots & 0 & 0\\
b_{k_1} & 2-2r_1^2 & b_{k_1} & \dots & 0 & 0\\
0 & b_{k_2} & 2-2r_2^2 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & \dots & b_{k_{N_x-1}} & 2-2r_{k_{N_x-1}}^2 & b_{k_{N_x-1}}\\
0 & 0 & \dots & 0 & 0 & 0
\end{pmatrix},
$$

$$
B_3 = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0
\end{pmatrix}.
$$

Denote by C the square matrix

$$
C = \begin{pmatrix} B_1 \\ B_2 B_1 + B_3 \end{pmatrix}
$$

and put

$$M = \left(C^{(N_t-1)/2}_{[N_x+2]}, C^{(N_t-1)/2}_{[N_x+3]}, \dots, C^{(N_t-1)/2}_{[2N_x+2]}\right)^\top,$$

where $C^{(N_t-1)/2}_{[i]}$ is the ith row of the matrix $C^{(N_t-1)/2}$.

Theorem 3. *The inverse problem* (24), (25) *is reducible to a system of linear algebraic equations $A_{ll}(k)q_k = F_k$, $l = N_x+1$, with the matrix $A_{ll}(k)$ of the form $A_{ll}(k) = MP$, where*

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & h_t & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & h_t & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The dimension of P is equal to $(2N_x+1)\times(N_x+1)$. The data vector is specified by the formula $F_k = F^{(2)}_k - MKF^{(1)}_k$, where $F^{(p)}_k = \left(f^{(p)}_{k_0}, f^{(p)}_{k_1}, \dots, f^{(p)}_{k_{N_x}}\right)^\top$, $p = 1, 2$, and

$$K = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ b_{k_1}/2 & 1-r_1^2 & b_{k_1}/2 & \dots & 0 & 0 \\ 0 & b_{k_2}/2 & 1-r_2^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b_{k_{N_x-1}}/2 & 1-r^2_{N_x-1} & b_{k_{N_x-1}}/2 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

3.2. Singular Values of the Matrix $A_{ll}(k)$

Here we study the behavior of singular values of $A_{ll}(k)$.

In the case of constant coefficient, the singular values of $A_{ll}(k)$ (Fig. 2) decrease in much the same way as those of the operator $A(k)$ (see Fig. 1).

Suppose now that the bottom topography is described by the function

$$H(x) = \left[g\left(\frac{9\beta}{10\pi^2}x^2 - \beta\right)^2\right]^{-1}, \qquad \beta = 5.$$

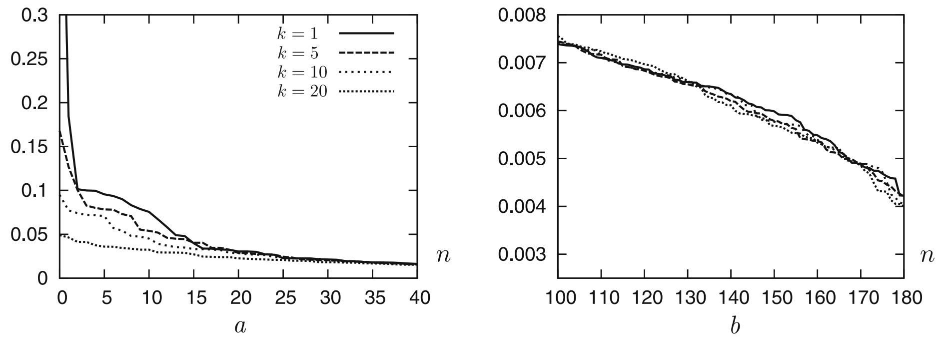


Fig. 2. Singular values $\sigma_n(A_{ll}(k))$ for $k = 1, 5, 10, 20$, $T = 3$, and $c = 1$: (a) $n = 0, \dots, 40$; (b) $n = 100, \dots, 180$

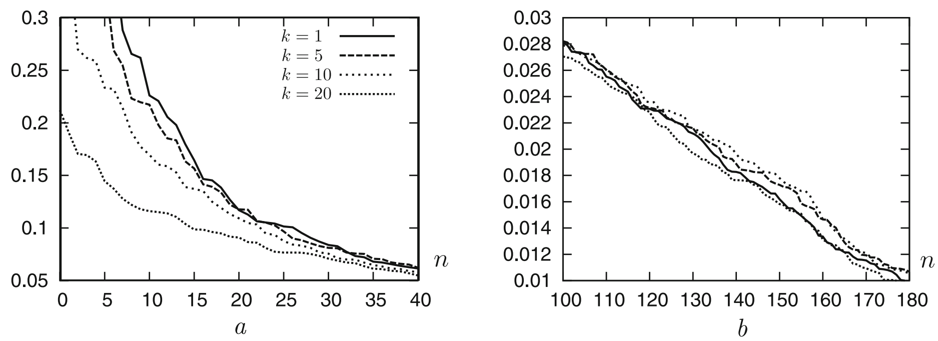


Fig. 3. Singular values $\sigma_n(A_{ll}(k))$ for $k = 1, 5, 10, 20$, $T = 3$, and $c = c(x)$: (a) $n = 0, \dots, 40$; (b) $n = 100, \dots, 180$

Then $c(x) = \left(\dfrac{9\beta}{10\pi^2}x^2 - \beta\right)^{-1}$. We calculate the singular values of $A_{ll}(k)$ in this case and display their behavior in Fig. 3.

3.3. An Algorithm for Constructing a Normal Pseudosolution and an r-Solution

Consider a system of linear algebraic equations $Aq = f$, where $A \in \mathbb{R}^m \times \mathbb{R}^n$ is a rectangular matrix, while $q \in \mathbb{R}^n$ and $f \in \mathbb{R}^m$ are vectors. Suppose, for example, that $m < n$. By the theorem on singular value decomposition, there exist orthogonal matrices $U \in \mathbb{R}^m \times \mathbb{R}^m$ and $V \in \mathbb{R}^n \times \mathbb{R}^n$ together with a nonincreasing sequence of non-negative numbers σ_j, $j = 1, \dots, m$, such that $A = U\Sigma V^\top$, where Σ is a rectangular matrix of the form

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \sigma_m & 0 & \dots & 0 \end{pmatrix}.$$

Rewrite the system $Aq = f$ in the form $U\Sigma V^\top q = f$.

Put $z = V^\top q$. Then $q = Vz$. Taking into account that $U^\top = U^{-1}$, we observe that $\Sigma z = U^\top f = g \in \mathbb{R}^m$. Hence, $z_j = g_j/\sigma_j$ if $\sigma_j \neq 0$. In the case when $\sigma_j = 0$ or $j > m$, we specify $z_j = 0$. Thereby we have constructed a normal pseudosolution $q_{np} = Vz$ of the problem $Aq = f$ [26].

Note that if the matrix A has relatively small singular values then large errors may appear in calculating the corresponding $z_j = g_j/\sigma_j$. To avoid the error accumulation, it is necessary to provide an operation of zeroing out small singular values. It is most natural to equate σ_j to zero starting with some number. Let Σ_r denote the matrix obtained from Σ by putting $\sigma_j = 0$ for $j = r+1, \dots, m$. The change from the original system $Aq = U\Sigma V^\top q = f$ to the system $U\Sigma_r V^\top q_r = f$ can be treated as a regularization with the parameter r. The solution q_r of the latter system is called the *r-solution* of the problem $Aq = f$. As usual, how to choose the parameter r is an important question. Indeed, if r is too small, the regularized system differs considerably from the original one; but if r is large then the calculated solution may have a very large error.

In the following theorem we show that the optimal value of r is chosen depending on the error ε in the data of the problem:

Theorem 4. *Let $A : Q \to F$ be a compact linear operator, and let Q and F be separable Hilbert spaces. If q_r^ε is an r-solution of the problem $Aq = f^\varepsilon$ with $|f - f^\varepsilon| \le \varepsilon$ then*

$$\left\|q_{np} - q_r^\varepsilon\right\|^2 \le \frac{b^2}{3r^3} + \varepsilon^2 \sum_{j=1}^{r} \frac{1}{\sigma_j^2}$$

and the optimal r satisfies the equation

$$-b^2/r^4 + \varepsilon^2\left(1/\left(\sigma_r^2\right) - 1/\left(\sigma_1^2\right)\right) = 0,$$

where $b = \|q_{np}\|$.

Proof. As known [26], the normal pseudosolution q_{np} of the problem $Aq = f$ admits the following series expansion:

$$q_{np} = \sum_{j=1}^{\infty} q_j v_j = \sum_{j=1}^{\infty} \frac{\langle f, u_j\rangle}{\sigma_j} v_j. \tag{26}$$

Here q_j are the Fourier coefficients of the function q_{np}, $\{\sigma_j\}$ is a nonincreasing sequence of singular values of the operator A, while $\{v_j\}$ and $\{u_j\}$ are the corresponding right- and left-singular vectors, respectively (these are the orthonormal sequences of functions).

Let q_r be an r-solution of the problem $Aq = f$; that is, its first r components coincide with those of q_{np} and all next are zeroes. Then

$$\left\|q_{np} - q_r^\varepsilon\right\|^2 \le \|q_{np} - q_r\|^2 + \left\|q_r - q_r^\varepsilon\right\|^2. \tag{27}$$

Without loss of generality, we assume that

$$q_{np}(x) = \sum_j q_j e^{ijx}, \qquad q_{np}(x) \in C^2[0, L].$$

Then there holds $|q_j| \le b/|j|^2$, where $b = \|q_{np}\|_{C^2[0,L]}$.

Thus, the first term in (27) is estimated as follows:

$$\|q_{np} - q_r\|^2 \le \sum_{j=r}^{\infty} q_j^2 \le \sum_{j=r}^{\infty} \frac{b^2}{j^4}.$$

Using the Cauchy integral test, we obtain

$$\sum_{j=r}^{\infty} \frac{b^2}{j^4} \le b^2 \int_r^{\infty} \frac{1}{x^4} dx = \frac{b^2}{3r^3}.$$

For the second term in (27) we obtain

$$\|q_r - q_r^\varepsilon\|^2 = \sum_{j=1}^{r} \frac{\langle f - f^\varepsilon, u_j\rangle^2}{\sigma_j^2} \le \sum_{j=1}^{r} \frac{\varepsilon^2}{\sigma_j^2}.$$

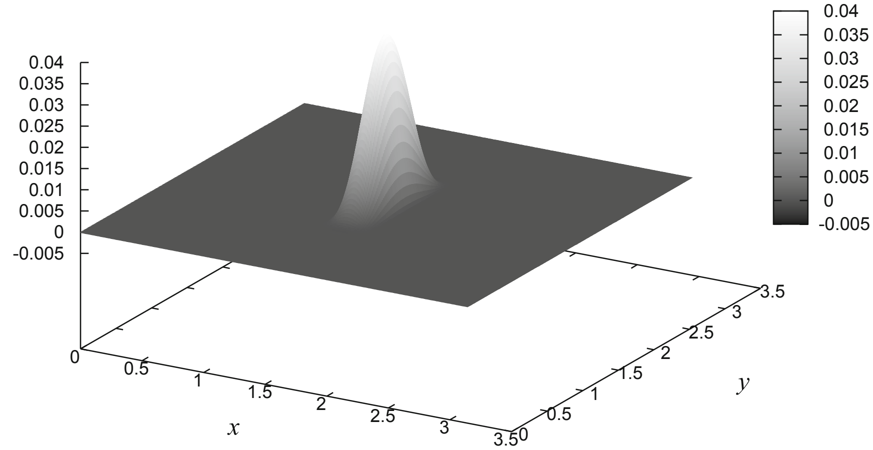


Fig. 4. Exact solution $q_e(x, y)$

Therefore,

$$\|q_{np} - q_r^\varepsilon\|^2 \leq \frac{b^2}{3r^3} + \varepsilon^2 \sum_{j=1}^{r} \frac{1}{\sigma_j^2}. \tag{28}$$

Evidently, the optimal number r is to minimize the right-hand side of inequality (28); that is,

$$r = \min_{l \in \mathbb{N}} \left(\frac{b^2}{3l^3} + \varepsilon^2 \int_1^l \frac{dx}{\sigma^2(x)} \right).$$

Hence, r is a solution of the equation

$$-b^2/r^4 + \varepsilon^2 \Big(1/(\sigma_r^2) - 1/(\sigma_1^2)\Big) = 0,$$

which proves the theorem. □

4. NUMERICAL EXPERIMENTS

Let the inverse problem (22), (23) have an exact solution of the form

$$q_e(x, y) = \sum_{k=1}^{N} q_{ek}(x) \sin ky, \qquad x \in \left(\frac{7\pi}{16}, \frac{9\pi}{16}\right), \qquad y \in \left(\frac{3\pi}{10}, \frac{7\pi}{10}\right), \tag{29}$$

where

$$q_{ek} = \frac{2}{\pi} \int_0^\pi \frac{\cos 16x + 1}{20} \frac{\sin 5y + 1}{5} \sin ky \, dy.$$

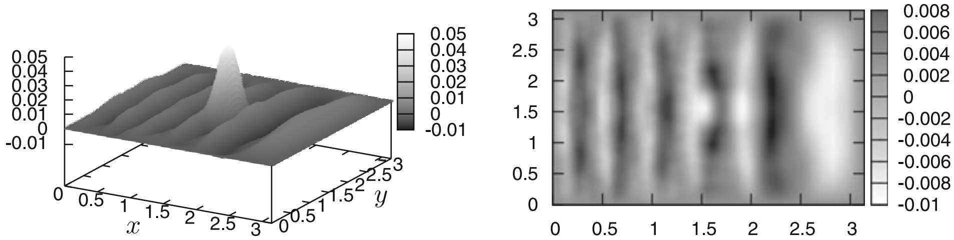
The exact solution (29) is plotted in Fig. 4.

As the known function $f^{(1)}(x, y)$ of problem (22) we took

$$f^{(1)}(x, y) = \frac{\cos 16x + 1}{20} \frac{\sin 5y + 1}{20}, \qquad x \in \left(\frac{7\pi}{16}, \frac{9\pi}{16}\right), \qquad y \in \left(\frac{3\pi}{10}, \frac{7\pi}{10}\right).$$

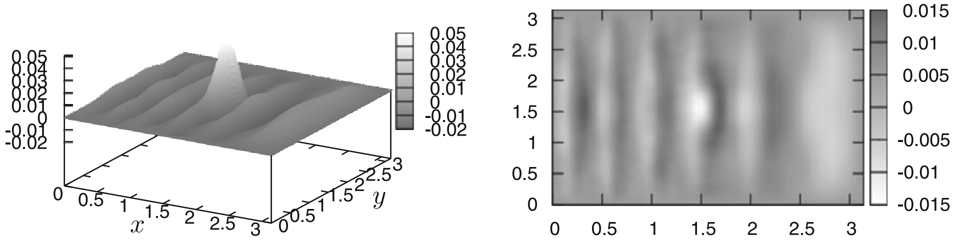
In numerical experiments, we considered the inverse problems (20), (21) for $L = \pi$ and $T = 3$ on the uniform grid with $N_x = 250$, $N_y = 200$, and $N_t = 591$ and solved the corresponding discrete problems (24), (25) for every $k = 1, 2, \ldots, 50$ using the singular value decomposition.

Fig. 5 illustrates the recovered rate of the seafloor displacement $q_r^\varepsilon(x, y)$ and its deviation from the exact solution $q_e(x, y)$.



a *b*

Fig. 5. The r-solution $q_r^\varepsilon(x, y)$ for $\varepsilon = 30\%$ and $r = 198$ (a); the difference between the exact solution $q_e(x, y)$ and the r-solution $q_r^\varepsilon(x, y)$ (b)



a *b*

Fig. 6. The reconstructed solution $q_{500}(x, y)$ obtained with the Landweber algorithm after 500 iterations (a); the difference between $q_{500}(x, y)$ and the exact solution $q_e(x, y)$ (b)

4.1. The r-Solution as an Initial Approximation in the Landweber Iteration

In [22], we used the gradient methods to solve the inverse problem (22), (23). Here we present the numerical results of successive application of the singular value decomposition and the Landweber iteration method to the perturbed problem (22), (23) with $L = \pi$, $T = 3$, and the data error $\varepsilon = 30\%$. Indeed, we took, as an initial approximation q_0, the r-solution $q_r^\varepsilon(x, y)$ (see Fig. 5) and computed the subsequent approximations $q_i(x, y)$ using the Landweber iteration. The approximate solution $q_{500}(x, y)$ generated after 500 iterations is displayed in Fig. 6, as is the difference between $q_{500}(x, y)$ and the exact solution $q_e(x, y)$.

Note that the r-solution has not improved upon applying the optimization (compare Figures 5, *b* and 6, *b*).

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